

Halperin (m, m', n) bilayer quantum Hall states on thin cylinders

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The Halperin (m, m', n) bilayer quantum Hall states are studied on thin cylinders. In this limit, charge density wave patterns emerge that are characteristic of the underlying quantum Hall state. The general patterns are worked out from a variant of the plasma analogy. Torus degeneracies are recovered, and for some important special cases a connection to well-known spin chain physics is made. By including interlayer tunneling, we also work out the critical behavior of a possible phase transition between the (331) state and the non-abelian Moore-Read state in the thin cylinder limit.

A very successful strategy to solve problems in condensed matter physics is to identify the correct simple state that a given interacting quantum system can be smoothly evolved into. On the other hand, our understanding of fractional quantum Hall states is based on radically new techniques[1]. Recently, however, a particular way has been discussed to adiabatically transform fractional quantum Hall liquids into trivial charge density wave (CDW) states. In Refs. 2, 3, 4, 5 it was observed that for certain Hamiltonians, a quantum Hall ground state does not undergo a phase transition when the two-dimensional surface of the system is deformed into a quasi one-dimensional (1D) limit, e.g. a thin torus or cylinder. In this limit, simple CDW patterns emerge[6]. These patterns are characteristic of the underlying quantum Hall state, and can serve as “labels” for the ground states and their elementary excitations. Despite their seeming simplicity, these labels turn out to be very efficient book-keeping tools. They give rise to simple and picturesque explanations for some abstract concepts in quantum Hall physics, such as the existence of fractional charges and topological sectors. Braiding statistics can also be derived in this language, at least for abelian states[7]. Moreover, the efficiency of these 1D labels in deriving various counting formulas for quasi-hole type states has been suggested early on by Haldane[8], who arrived at the same effective language by different means[8, 9]. While such formulas can also be obtained by other methods [10, 11], Read has recently demonstrated agreement between the latter and the 1D approach[10]. So far, however, the language of these 1D labels has been discussed only for single component quantum Hall states. Here we extend this language to the entire class of Halperin (m, m', n) bilayer states[12]. To this end, we develop a new technique that allows the extraction of the general thin cylinder limits of these states directly from the many-body wavefunctions. Special attention is paid to the class of (m, m, m) states, and to the (331) state. For the latter, implications concerning the possibility of a continuous phase transition into a Moore-Read[13] (Pfaffian) state are discussed.

Reduction to a 1D electrostatic lattice problem. In Refs. 2, 3, 5, pseudo-potential Hamiltonians

have been used to determine the thin torus limits of specific quantum Hall states. This method proves to be unwieldy for a class as general as the (m, m', n) states, as it would require high orders in perturbation theory for large m, m', n . Here we derive a general method to extract the limiting CDW patterns directly from many-body wavefunctions. Unless otherwise noted, we assume $m, m' > n$. The (m, m', n) states are expected to be $d = mm' - n^2$ fold degenerate on the torus[14]. Since the torus versions of the (m, m', n) wavefunctions are quite complicated, we work on the cylinder, where the same CDW patterns must appear between the edges as the circumference approaches zero. A straightforward generalization of Halperin’s (m, m', n) state to cylindrical topology leads to the expression

$$\psi_{mm'n} = \left(\prod_I \xi_I^s \right) Q(\{\xi_i\}; \{\xi_I\}) \times \exp\left(-\frac{1}{2} \sum_{\alpha} x_{\alpha}^2\right) \quad (1)$$

where $Q(\{\xi_i\}; \{\xi_I\}) = \prod_{i < j} (\xi_i - \xi_j)^m \prod_{I < J} (\xi_I - \xi_J)^{m'} \prod_{i, J} (\xi_i - \xi_J)^n$. For $s = 0$, the polynomial part of Eq. (1) is thus the same that would appear in the usual disk topology, with the particle coordinates $z_{\alpha} = x_{\alpha} + iy_{\alpha}$ replaced by functions $\xi_{\alpha} = \exp(\kappa z_{\alpha})$ periodic in y_{α} , where $\kappa = 2\pi/L_y$ and L_y is the circumference of the cylinder. We assume Landau gauge, $A = (0, x)$, for the vector potential, letting the magnetic length equal to one. Uppercase (lowercase) indices refer to particles in the upper (lower) layer, whereas Greek indices refer to both layers. The integer s in Eq. (1) can be interpreted as placing s quasiholes in the upper layer at $x = -\infty$. This does not affect the local properties of the incompressible fluid described by Eq. (1), which is confined to a finite “ribbon” on the infinite cylinder. It is left understood that each such ribbon on the cylinder would be infinitely degenerate by translation. We will see, however, that precisely d distinct CDW patterns are generated in the interior of the ribbon when the thin cylinder limit is taken, and when s runs through all possible values and translational symmetry is used.

We will now compute the thin cylinder limit of Eq. (1) using the following strategy. The lowest Landau level (LLL) on the cylinder has a natural basis of ring shaped

orbitals $\psi_n = \xi^n \exp(-x^2/2 - \kappa^2 n^2/2)$ localized at a height $x \sim \kappa n$ of the cylinder and delocalized around the y circumference. We may expand the state Eq. (1) in product wavefunctions $\psi_{n_1 \dots n_N} = \psi_{n_1}(x_1) \dots \psi_{n_N}(x_N)$ where the N particles occupy definite LLL orbitals. As explained in Ref. 6, the coefficients in this expansion are of the form $\exp(\frac{1}{2}\kappa^2 \sum_{\alpha} n_{\alpha}^2) \cdot C(n_1 \dots n_N)$, where $C(n_1 \dots n_N)$ is the coefficient of the monomial $\xi_1^{n_1} \dots \xi_N^{n_N}$ in the polynomial part of Eq. (1). From this it is clear that those products $\psi_{n_1 \dots n_N}$ in the expansion of Eq. (1) will dominate as $\kappa \rightarrow \infty$ for which the quantity

$$S = \sum_{\alpha=1}^N n_{\alpha}^2 \quad (2)$$

is maximal. Since the single particle orbitals ψ_n are well separated along x as $\kappa \rightarrow \infty$, the dominating products $\psi_{n_1 \dots n_N}$ describe a well defined CDW pattern in the thin cylinder limit. To identify this pattern, we observe that for any monomial $\xi_1^{n_1} \dots \xi_N^{n_N}$ that has a non-zero coefficient in the polynomial $(\prod_i \xi_i^s) Q(\{\xi_i\}; \{\xi_I\})$, the exponents n_{α} satisfy the following relation:

$$n_{\alpha} = \sum_{\beta} (m_{\alpha\beta} + p_{\alpha\beta}) + s\delta_{\alpha,\uparrow} \quad (3)$$

where $m_{\alpha\beta}$ equals $m/2$ ($m'/2$) if both indices are in the lower (upper) layer, and $n/2$ otherwise, $p_{\alpha\beta}$ is an *antisymmetric* matrix whose values are restricted to $\{-m_{\alpha\beta}, -m_{\alpha\beta}+1, \dots, m_{\alpha\beta}\}$, and we define $\delta_{\alpha,\uparrow} \equiv \sum_I \delta_{\alpha,I}$, $\delta_{\alpha,\downarrow} \equiv \sum_i \delta_{\alpha,i}$. In Eq. (3), each choice for a particular value of $p_{\alpha\beta}$ corresponds to the choice of the term $\xi_{\alpha}^{m_{\alpha\beta}+p_{\alpha\beta}} \xi_{\beta}^{m_{\alpha\beta}-p_{\alpha\beta}}$ in the expansion of the factor $(\xi_{\alpha} - \xi_{\beta})^{2m_{\alpha\beta}}$ in $Q(\{\xi_i\}; \{\xi_I\})$. We find that the choice of $p_{\alpha\beta}$ that maximizes Eq. (2) is always of the form

$$p_{\alpha\beta} = m_{\alpha\beta} \operatorname{sgn}(\sigma_{\alpha} - \sigma_{\beta}), \quad (4)$$

where σ is a permutation of N objects. A proof of this statement can be found in Ref. 15. Figuratively speaking, Eq. (4) says that each solution of the maximization problem Eq. (2) corresponds to a “ranking” of the N particles by means of the permutation σ . If we imagine the N particles arranged in a row according to their ranking, we may call such an arrangement a state of “the squeezed lattice”. The term “squeezed” alludes to the fact that, in contrast to the “real space” (orbital) arrangement of the particles in the thin cylinder limit that we are seeking, there are no empty sites in this squeezed lattice. However, we will see below that the particle arrangements on the squeezed lattice and in real space are closely related. Via Eq. (4), the problem is reduced to finding a permutation σ that maximizes S . Since particles in the same layer are identical, S is invariant under permutations of such identical particles on the squeezed lattice. For our purposes, the state of the squeezed lattice is thus fully described by the data $k_{\alpha} = \delta_{\bar{\sigma}_{\alpha},\uparrow} - \delta_{\bar{\sigma}_{\alpha},\downarrow}$, which equal $+1$

or -1 depending on whether the particle occupying the α -th site, $\bar{\sigma}_{\alpha} \equiv \sigma^{-1}(\alpha)$, belongs to the upper or lower layer. It turns out that in terms of the k_{α} , S can be written as $S = -\frac{1}{4}n(m+m'-2n)E$, where

$$E = \sum_{\alpha < \beta} -(\beta - \alpha)q_{\alpha}q_{\beta} - 2Q_0 \sum_{\alpha} \alpha q_{\alpha} + \text{const.} \quad (5)$$

and $q_{\alpha} = k_{\alpha}(1 + k_{\alpha}\Delta)$, $\Delta = (m' - m)/(m + m' - 2n)$, $Q_0 = (2s + N^{\downarrow}(n - m) + N^{\uparrow}(m' - n) + m - m')/(m + m' - 2n)$, and we write N^{\downarrow} (N^{\uparrow}) for the number of particles in the lower (upper) layer. Eq. (5) can be interpreted as an electrostatic energy assigned to each configuration of the squeezed lattice, where a charge $q^{\uparrow,\downarrow} = \pm(1 \pm \Delta)$ is assigned to upper and lower layer particles, respectively. The first term in Eq. (5) is a linear 1D Coulomb interaction between particles. The second term can be interpreted as a linear potential due to an external charge distribution, e.g. a charge $+Q_0$ at the left boundary and a charge $-Q_0$ at the right boundary of the system. Once the charge configuration minimizing E is found, the corresponding real space configuration, i.e. the orbital position n_{α} of the α -th particle, follows from an “un-squeezing” rule obtained from Eqs. (3), (4)

$$n_{\alpha} = \begin{cases} m'N_{\alpha}^{\uparrow} + nN_{\alpha}^{\downarrow} + s & \text{if } k_{\alpha} = +1 \\ mN_{\alpha}^{\downarrow} + nN_{\alpha}^{\uparrow} & \text{if } k_{\alpha} = -1, \end{cases} \quad (6)$$

where N_{α}^{\uparrow} (N_{α}^{\downarrow}) is the number of particles with $k_{\alpha'} = +1$ ($k_{\alpha'} = -1$) to the left of site α on the squeezed lattice, i.e. for $\alpha' < \alpha$. Eq. (5) can be viewed as a discrete version of the plasma analogy in the thin cylinder limit.

Solution of the electrostatic problem. We want to minimize the energy Eq. (5) for given parameters m , m' , n , and s . It is useful to write $m - n = gr$, $m' - n = gr'$, where r and r' are coprime integers. Since there are no vacancies on the squeezed lattice, the particle configuration and thus the energy E is fully determined by specifying the positions of the lower layer particles alone. If α_j is the position of the j -th lower layer particle on the squeezed lattice, we find

$$E = q^{\uparrow}(q^{\uparrow} - q^{\downarrow}) \sum_{j=1}^{N^{\downarrow}} (\alpha_j - x_j)^2 + \text{const}, \quad (7)$$

$$x_j = \frac{r + r'}{r'}j - \frac{s}{gr'} - \frac{r}{r'} + \frac{1}{2}. \quad (8)$$

It is now clear how to minimize E and thus maximize S . We position the lower layer particles as closely as possible to the minima x_j , subject only to the constraint that the positions α_j are integer, and that $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_{N^{\downarrow}} \leq N$. This determines a pattern of lower and upper layer particles in squeezed space. Certain boundary effects may be present as follows. For example, the equilibrium position of the first lower layer particle, x_1 , may be negative. Then for a range of j values, the x_j 's

will be inaccessible by the respective particle coordinates α_j . This will lead to a clustering of lower layer particles (q^\downarrow charges) at the left boundary of the squeezed lattice. This in turn can be interpreted as a screening cloud to screen the charge $+Q_0$ discussed below Eq. (5). However, for sufficiently large particle number j there will be a regime where the particle position α_j can be placed within a distance $1/2$ or less from x_j . This regime of the squeezed lattice we will call the "bulk" as opposed to the screening cloud on either end of the squeezed lattice. The appearance of different regimes on the cylinder is expected for the general wavefunction Eq. (1), not only in the thin cylinder limit. In particular, one may consider Eq. (1) with N^\uparrow/N^\downarrow much different from the correct ratio $\nu^\uparrow/\nu^\downarrow$ of the upper and lower layer filling factors in the (m, m', n) phase, where $\nu^\uparrow = (m-n)/d$ and $\nu^\downarrow = (m'-n)/d$, respectively. In this case Eq. (1) is known to describe a phase separated state with phase boundaries between an (m, m', n) bilayer phase and a Laughlin-type phase in either the upper or lower layer [16]. It is the bulk region of the squeezed lattice which corresponds to the thin cylinder CDW pattern of the (m, m', n) phase when the squeezed lattice is transformed into real space via the un-squeezing rules Eq. (6). We now derive some properties of these patterns. First we note that some of the x_j may be half-odd integer. In this case both values $\alpha_j = x_j \pm 1/2$ lead to a minimum of E . The thin cylinder limit of the wavefunction Eq. (1) is an equal amplitude superposition of the states generated by all these configurations. In squeezed as well as in real space, the patterns corresponding to the two cases $\alpha_j = x_j \pm 1/2$ differ by an exchange between two adjacent particles in different layers. For example, for $m = 5$, $m' = 3$ and $n = 2$ with $s = 0$, we find the following real space thin torus pattern from the procedure described above:

$$\text{X0X00} \uparrow 00 \uparrow 00\text{X0X00} \uparrow 00 \uparrow 00 \dots \quad (9)$$

where characters denote the states of consecutive LLL orbitals, 0 meaning unoccupied, and X0X represents a triplet $\uparrow 0 \downarrow + \downarrow 0 \uparrow$. The unit cell of the pattern Eq. (9) has 11 sites. Using translational symmetry, this fully accounts for the ground state degeneracy of the (532) state on a torus. The general situation is slightly more complicated. From Eq. (8), we observe that when $j \rightarrow j + r'$, x_j increases by the integer $r + r'$, and the pattern in squeezed space will repeat itself. The squeezed space unit cell thus has size $r + r'$. It consists of r' charges q_\downarrow and r charges q_\uparrow and is thus neutral. The size of the unit cell in real space can then be obtained from Eq. (6) with $N_\alpha^\uparrow \rightarrow N_\alpha^\uparrow + r$, $N_\alpha^\downarrow \rightarrow N_\alpha^\downarrow + r'$. It is easy to see that both lines in Eq. (6) yield $n_\alpha \rightarrow n_\alpha + (mm' - n^2)/g$. The real space unit cell of the thin cylinder pattern thus has size d/g . Note that from this we correctly obtain $\nu^\uparrow = rg/d$, $\nu^\downarrow = r'g/d$. On a (thin) torus, a given pattern thus only accounts for a fraction $1/g$ of the full degeneracy, using translational symmetry. However, we see from Eq. (6)

that varying s will in general yield different real space patterns. By "different", we mean that the patterns are not related by translation. The patterns only repeat, up to a translation, when $s \rightarrow s + g$. To see this, we note that one can find integers a, b such that $ar = r'b + 1$. If we let $s \rightarrow s + g$, $j \rightarrow j + a$ in Eq. (8), we find that x_j is shifted by the integer amount $a + b$. This means that the bulk pattern on the squeezed lattice is merely shifted by a constant. It is then easy to show from Eq. (6) that the real space bulk pattern is also just shifted by a constant. By varying s , we thus generate g different bulk CDW patterns in the thin cylinder limit, each having a unit cell of size d/g . On the torus these patterns, and those related by translation, correspond to $g(d/g) = d$ distinct ground states, as expected. We see that for $g > 1$, i.e. whenever $m' - n$ and $m - n$ are not coprime, not all of the ground states on the torus are related by translation. Furthermore, one may show from Eq. (8) that spin fluctuations as discussed for Eq. (9) are always present in precisely one of the g ground state patterns. In contrast, charge is always frozen in the thin cylinder limit, and hence these fluctuations are unique to multi-component systems. In the following we discuss some cases of special interest.

(m, m, m) states. For $m = m' = n$ the quantity S , Eq. (2), does not depend on the permutation σ , Eq. (4). Every configuration of upper and lower layer particles on the squeezed lattice, for N^\uparrow, N^\downarrow fixed, has the same amplitude in the thin cylinder wavefunction. The corresponding real space configurations obtained from Eq. (6) live on a diluted lattice where every m -th LLL-orbital is occupied. The thin cylinder wavefunction is thus an equal amplitude superposition of states on the diluted lattice with given N^\uparrow, N^\downarrow . If the layer index is regarded as spin-1/2 index, these thin cylinder wavefunctions are ground states of the ferromagnetic Heisenberg Hamiltonian $-J \sum_j (S_{mj} \cdot S_{(m+1)j} - 1/4)$. Using degenerate perturbation theory, we have verified that the pseudo-potential Hamiltonian whose exact ground state is the (m, m, m) state (see, e.g., Ref.17) does indeed assume this form in the thin cylinder limit, to leading order in $\exp(-\kappa^2)$. This shows that the low-energy sector of the (m, m, m) pseudo-potential Hamiltonian has a gapless quadratically dispersing branch even in this limit. By adding a generic perturbation that breaks the $SU(2)$ symmetry down to an easy-plane symmetry, the dispersion of this mode becomes linear, as is well known from the theory of spin-1/2 chains. Thus, the low energy spectrum we obtain in the thin cylinder limit completely agrees with the spectrum expected for (m, m, m) states on infinite two-dimensional (2D) surfaces[18].

The (331) state. According to the general framework established above, the eight (331) ground states on the torus come in two classes of four states. These classes are distinct by the fact that their respective members evolve into one of the following two CDW patterns in the thin torus limit, up to translations:

$$\downarrow 0 \uparrow 0 \downarrow 0 \uparrow 0 \downarrow 0 \uparrow 0 \downarrow 0 \uparrow 0 \downarrow 0 \uparrow 0 \quad (10a)$$

$$XX \ 00 \ XX \ 00 \ XX \ 00 \ XX \ 00 \ XX \ 00 \quad (10b)$$

where again XX stands for $\uparrow\downarrow + \downarrow\uparrow$. We have shown that indeed both these states remain zero energy eigenstates of the (331) pseudo-potential Hamiltonian (see, e.g., Ref. 17) in the thin torus limit when leading corrections in $\exp(-\kappa^2)$ are considered. As expected (c.f. Refs. 2, 3, 4, 5), it is easy to show that one may form domain walls between the two patterns in Eq. (10) that carry the correct fractional charge quantum numbers for Laughlin quasi-holes in the (331) state, e.g. $+1/8$ in the upper layer and $-3/8$ in the lower. There is an intimate connection between the wavefunctions of the (331) state and the $\nu = 1/2$ Pfaffian state[19]. This is particularly suggestive if we regard the thin torus CDW-patterns as the "labels" for the respective ground states, as discussed initially. For, if we drop the spin index in the (331)-patterns Eq. (10) by simply writing 1 for each occupied site, we obtain 10101010... for Eq. (10a) and 11001100... for Eq. (10b). These are just the labels associated with the $\nu = 1/2$ Pfaffian[5]. Note that the latter is only sixfold degenerate on the torus, as the four patterns related to Eq. (10a) by translation collapse into mere two patterns when spin indices are dropped. The question of the possibility and nature of a continuous (phase) transition between the (331) state and the Pfaffian has generated much interest[20, 21]. It is instructive to study this phase transition in the thin torus limit. To this end, we examine the effect of a small interlayer tunneling term H_t added to the pseudo-potential Hamiltonian of the (331) state. We first consider the fate of the ground state Eq. (10a) when H_t is added. The matrix elements of H_t can be regarded as spin flips. Writing $H = H_0 + H_1 + H_t$, where H_0 contains all terms of the pseudo-potential Hamiltonian acting between identical or nearest-neighbor LLL-orbitals, H_1 all remaining terms, we can treat $H_1 + H_t$ in degenerate perturbation theory in the thin torus limit. Both states in Eq. (10) are ground states of H_0 , as well as all states related to Eq. (10a) by spin flips. In the sector spanned by states of the latter kind, we find that to leading order in $\exp(-\kappa^2)$, the effective Hamiltonian is given by $H_{eff} = J_z \sum_j \sum (\sigma_{2j}^z \sigma_{2j+2}^z + 1) + t \sum_j \sigma_{2j}^x$, where the $\sigma_{2j}^{x,z}$ are Pauli matrices, and $J_z \sim \exp(-2\kappa^2)$. The hopping parameter t is assumed to be much less than $\exp(-\kappa^2/2)$ (the gap of H_0) but may be much larger than J_z . H_{eff} is a transverse field Ising model which has a well-known phase transition at $t = J_z$. As a result of this transition, the two ground states descended from Eq. (10a) and its spin flipped counterpart are replaced by a single ground state in the large t phase. The same happens in the ground state sector related to Eq. (10a) by a single site translation. No transition is found in the sector related to ground states of the type Eq. (10b). It was already observed in Ref. 3 that ground states in different

sectors will in general respond differently to a perturbation of the pseudo-potential Hamiltonian in the thin torus regime. In this regime, the degeneracy of the two types of ground states shown in Eq. (10) is not maintained as t increases. Yet if we follow these ground states individually, at large t six of the eight ground states evolve exactly into the six thin torus ground states of the $\nu = 1/2$ Pfaffian, with spins polarized along the x direction. It is so far unclear to what extent the behavior described here can be extrapolated to the regime of 2D quantum Hall phases. However, we argue that any complete theory of a phase transition between the (331) state and the Pfaffian should reproduce this behavior when the system is deformed into a thin torus. In this sense, an intimate relation between this "topological" phase transition and the transverse field Ising chain transition may be suspected.

Conclusion. We have derived a method to obtain the limiting charge-density-wave form of Halperin (m, m', n) bilayer states on thin cylinders/tori. We found that basic properties of these states such as filling factors, torus degeneracies and fractional charges can be obtained from a one-dimensional variant of the plasma analogy. In some important special cases, a connection to the well known physics of spin-1/2 chains has been made. This also leads to a possible connection between phase transitions in spin chains and in topologically ordered states of matter.

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